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Quantification of Modeling Uncertainty in Aeroelastic Analyses

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Traditional uncertainty quantification techniques in engineering analysis concentrate on the quantification of parametric uncertainties: inherent natural variations of the input variables. In problems with complex or newer modeling methodologies, the variabilities induced by the modeling process itself (known as model-form and predictive uncertainties) can become a significant source of uncertainty to the problem. This work demonstrates two model-form uncertainty quantification methods on an unsteady aeroelastic problem: Bayesian model averaging and the adjustment factors approach. While the Bayesian model averaging approach is more robust and has been shown to more completely quantify the total uncertainty, it also requires the presence of experimental data, which are not always readily available in preliminary design. As such, this work introduces an uncertainty quantification methodology for use in aeroelastic analysis that uses the modeling uncertainty to drive the necessity of further experimental data points. Within this methodology, the modified adjustment factors approach has been developed to calculate the sensitivity of the adjusted models to the model probability assumptions being input into the work, facilitating the flow of the design methodology.

I. Introduction

UNCERTAINTY in engineering analysis can originate from three sources: parametric uncertainty, predictive uncertainty, and model uncertainty [1]. While parametric uncertainty refers to the natural variability of the input parameters into a particular analysis, model-form and predictive uncertainties refer to the variability involved with the modeling process used within an analysis. Parametric uncertainty has been explored and quantified in depth for problems in the literature, but much less work has been done in the fields of model and predictive uncertainties. When simulation-based modeling is used in engineering problems, there are often multiple models available to represent a given situation. In well-understood phenomena, these models are refined and a best model will often emerge. However, in many multiphysics problems such as aeroelasticity, uncertainty exists regarding the best model to represent a physical problem, as well as the relative merit and accuracy of this best simulation. Because of this uncertainty, multiple aeroelastic programs or simulation models can produce different results for the same problem, depending on the aerodynamic and structural analyses used within the code or, specifically, the assumptions that go into forming the models. To fully use these simulation tools, the uncertainty among the models (model-form uncertainty) and the uncertainty between the models and the true physical scenario (predictive uncertainty) must be quantified to present an accurate representation of the metric of interest.

It is critical in this uncertainty quantification process to consider the availability and necessity of additional information. At a certain stage in a problem, the modeling uncertainty cannot be further reduced without the introduction of additional data, such as refined models or experimental validation points. This additional information is often difficult or expensive to obtain. Thus, it is crucial to identify the areas in the design methodology where such data would be most beneficial to the overall reduction of the modeling uncertainty. This work introduces a design methodology that, instead of performing a blanket amount of additional analyses, uses the modeling uncertainty itself to drive the necessity and location of additional data points in the problem.

In this work, an adjustment factors approach (AFA), first demonstrated as a methodology to use expert opinions in the Bayes theorem by Mosleh and Apostolakis [2], is initially used to quantify the model-form uncertainty introduced to the problem by disagreement between the models of the physical problem. In the implementation of this AFA, probabilities are assigned to the models of interest based on expert opinions. These model probabilities $P(M_i)$ are defined as the probability that a particular model is the exact representation of the physical problem. The value for model probabilities is thus bounded as $0 \leq P(M_i) \leq 1$, such that the sum of all model probabilities is equal to one. Once assigned, these model probabilities can then be used in the quantification of model-form uncertainty by implementing an AFA to develop a prediction of the parameter of merit that considers the uncertainty between the various models. However, the adjusted model created by this approach is mathematically dependent on the model probabilities, which are assigned through often incomplete prior knowledge of the problem or the relative fidelity or accuracy of the model. The sensitivity of the resulting adjusted model to the individual model probabilities is an important parameter to quantify. To calculate the sensitivity of the model to the individual model probabilities, the modified AFA was developed in this work to identify the sensitivity of the adjusted model to the model probabilities assigned to them. If the adjusted model is shown to be sensitive to the model probabilities input to the problem, further information regarding the models themselves might be required to reduce the uncertainty in the modeling.

While the AFA quantifies the model-form uncertainty within the problem, it is still possible for the predictive uncertainty to be the driving uncertainty source. If it is shown that the adjusted models are sensitive to the individual model probabilities, that indicates that the

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predictive uncertainty within the problem can be a considerable source of uncertainty. In this case, it becomes necessary to use additional methods to quantify and reduce this predictive uncertainty by including additional data, such as model-validation cases or full experimental prototyping of the proposed model. Bayesian model averaging (BMA) [3] can be implemented to quantify both the predictive and model-form uncertainties in the problem. By including the experimental data points, the predictive uncertainty can be quantified specifically through BMA, and the model-form uncertainty in the problem can be reduced through the implementation of Bayesian model updating to refine the model probability predictions.

In this paper, the developed design methodology is applied to an aeroelastic problem: a flutter analysis on a two-degree-of-freedom (2-DOF) airfoil subject to unsteady aerodynamics. The model-form uncertainty introduced to the problems by the multiple modeling techniques will first be quantified using the AFA. The model probabilities in this approach will be assigned based on expert opinions of the relative merit and accuracy of the models. The sensitivity of the adjusted model to the individual model probabilities will then be estimated by the modified AFA developed in this work. The combination of the uncertainty in the adjusted model, as well as the sensitivity of the individual model probabilities, can then serve to guide future refinement of the models: specifically, the necessity of additional experimental validation points for the models. If deemed necessary by the traditional and modified AFA, further refinement of the adjusted model and model probabilities can then be achieved through the introduction of experimental data points and the implementation of BMA and Bayesian model updating. In this paper, the full methodology implementation, from the traditional AFA to the BMA and updating, are demonstrated for the aeroelastic problem.

II. Modeling Uncertainty Definition

When a physical problem is represented as a model, uncertainties are inherent to the modeling process. In aeroelastic design, these uncertainties can arise from multiple sources, such as the input parameters of the model, the fidelity of aerodynamic analyses, and the aerodynamic and structural discretization of the physical domain. Extensive work has been done in the past on parametric uncertainty on structural inputs [4–6], aerodynamic inputs [5,7], and environmental loading conditions [8]. However, the full uncertainty associated with modeling consists of more than parametric input and includes other factors, such as the uncertainties introduced by the modeling process itself. At a high level, the total uncertainty resulting from modeling can be broken into three distinct components [1]: model-form uncertainty, parametric uncertainty, and predictive uncertainty (Fig. 1).

In Fig. 1, $\tilde{f}_i(\bar{x})$ represents both the model-form and parametric uncertainties in the problem and $\hat{\varepsilon}$ represents the predictive uncertainty. Any uncertainty in the input vector $\bar{x}$ is considered parametric uncertainty. While this uncertainty is propagated through the model, it is separable from model-form uncertainty in a well-understood problem. The function $\tilde{f}_i$ represents a model of the system. When multiple models are considered, the difference between the values of $\tilde{f}_i(\bar{x})$ is considered model-form uncertainty. Finally, the difference between the model's representation of the system $\tilde{f}_i(\bar{x})$ and the true value of the analysis y is called the predictive uncertainty $\hat{\varepsilon}$. The predictive uncertainty in a problem is a result of the assumptions made in the modeling process. Parametric uncertainty quantification methods and applications have been addressed in depth in the existing literature [4–12]; however, the other two sources of uncertainty (both model-form and predictive uncertainties) are frequently ignored in problems involving modeling.

A. Model-Form Uncertainty

When an analysis must be done for an engineering problem, a representative model is often constructed to allow for analysis of the system. To construct this model, assumptions regarding the system must be made to simplify the problem so a model can feasibly and efficiently be constructed. These assumptions often vary between models and modeling packages, resulting in multiple solutions to identical problems. The difference between multiple models of the same problem is representative of model-form uncertainty: the uncertainty induced by the disagreement among multiple models of the same phenomenon. Because the multiple models give different answers to the same problem, a method must be used to combine individual results into a unified solution while quantifying this solution's uncertainty induced by disagreement between the models. Multiple methods have been developed and implemented in the literature to quantify this model-form uncertainty, such as BMA [3], the AFA [2], ensemble BMA [13], and continuous model expansion [14].

The AFA was first demonstrated as a method to use expert opinions in the Bayes theorem by Mosleh and Apostolakis [2] in 1986. This method uses an adjustment factor to modify the result of the best model, which is defined as the model with the highest model probability among the model set being considered. The AFA has been demonstrated on multiple engineering problems by Zio and Apostolakis [15] and Reinert and Apostolakis [16].

The adjustment factor can be represented by multiple types of distribution, such as a normal or lognormal distribution, resulting in the use of different adjustment factors. In the additive AFA, the adjustment factor E_a^* is assumed to be a normal random variable. The representation of the adjusted model prediction is shown in Eq. (1), where y^* represents the best model based on expert opinion:

$$y = y^* + E_a^* \tag{1}$$

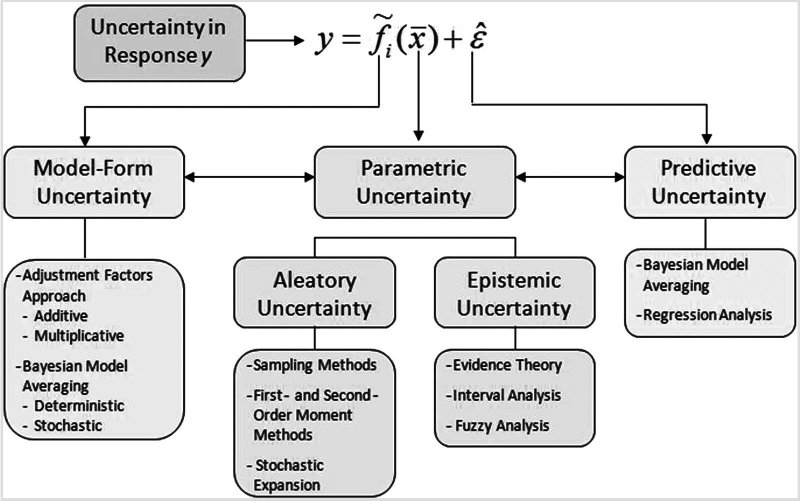

Fig. 1 Modeling uncertainty breakdown.

Knowing the results of multiple models, as well as their probabilities based on the provided expert opinion, the means and variances of the adjusted model can be calculated by Eqs. (2–5):

$$E(E_a^*) = \sum_{i=1}^{N} P(M_i)(y_i - y^*) \tag{2}$$

$$\mathrm{Var}\,(E_a^*) = \sum_{i=1}^{N} P(M_i)[y_i - E(y)]^2 \tag{3}$$

$$E(y) = y^* + E(E_a^*) \tag{4}$$

$$\mathrm{Var}\,(y) = \mathrm{Var}(E_a^*) \tag{5}$$

In the preceding equations, $E(y)$ represents the mean value of y, N is the total number of models considered, $P(M_i)$ represents the probability of model M_i based upon expert opinion, and y_i represents the prediction of model M_i. From the preceding equations, the mean and the standard distribution of the adjusted model y can be calculated and a normal distribution of the output can be constructed [Eqs. (4) and (5)].

In Eqs. (2) and (3), $P(M_i)$ represents the model probability of model i; by definition, it is the probability that model i is the best model among the model set considered. In the AFAs, this model probability is defined by expert opinions regarding the models of interest. As the model probabilities remain bounded by the laws of probability theory, constraints exist upon the values of the model probabilities, as shown in Eq. (6):

$$\sum_{i=1}^{N} P(M_i) = 1, \quad \text{such that } 0 \leq P(M_i) \leq 1 \tag{6}$$

While the additive AFA assumes a normally distributed adjusted model, additional distributions can also be assumed. For example, in the multiplicative AFA, the adjustment factor E_m^* is assumed to be a lognormal random variable. As such, the adjusted model prediction for the multiplicative AFA is shown in Eq. (7):

$$y = y^* * E_m^* \tag{7}$$

Assuming that the results and the probabilities of the multiple models are known, the means and variances of the natural log of the adjusted model and adjustment factor can be calculated by Eqs. (8–11):

$$E(\ln|E_m^*|) = \sum_{i=1}^{N} P(M_i)(\ln|y_i| - \ln|y^*|) \tag{8}$$

$$\mathrm{Var}\,(\ln|E_a^*|) = \sum_{i=1}^{N} P(M_i)[\ln|y_i| - E(\ln|y|)]^2 \tag{9}$$

$$E(\ln|y|) = \ln|y^*| + E(\ln|E_m^*|) \tag{10}$$

$$\mathrm{Var}\,(\ln|y|) = \mathrm{Var}(\ln|E_m^*|) \tag{11}$$

The AFA produces a statistical distribution of the adjusted model, accounting for the variation among the individual models. This distribution is dependent upon the expert opinion that goes into the model probabilities. Expert opinions are not infallible though; therefore, an additional degree of uncertainty is introduced to the final distribution due to the uncertainty surrounding the model probabilities.

The uncertainty in the model probabilities can lead to multiple problems with the AFA. Because of the weighting of the adjustment factors by the model probabilities, it is possible for the model probabilities to have significant effects on the adjusted model. If changes in model probabilities lead to large changes in the adjusted model, the adjusted model becomes very dependent on the model probabilities in addition to the variance among the models. Thus, it is critical to be able to identify the problems where the adjusted model is sensitive to the model probabilities. To identify these sensitivities, the modified AFA is developed in this work.

While the traditional AFA handles the model probabilities $P(M_i)$ as deterministic values, the modified AFA defines the individual model probabilities as normally distributed stochastic values, as shown in Eq. (12):

$$P(M_i) = N[P(M_i)_{\exp}, \sigma_i] \tag{12}$$

where

$$\sigma_i = \min[0.05, 0.25^* P(M_i)_{\exp}]$$

$P(M_i)_{\exp}$ represents the original model probability as defined based upon the expert opinions, and σ_i is a variance applied to the model probabilities in the approach. This variance is developed as a metric to explore the design space with respect to the model probabilities. As a result of the definition in Eq. (12), there are now N distributions of model probabilities for each of the N models that are then independently sampled using Monte Carlo sampling to obtain a set model probability values. Before these sampled values are used, though, they must be renormalized to maintain the constraints set forth in Eq. (6). This normalization involves scaling the values for the model probabilities such that the constraints in Eq. (6) (namely, that the summation of the model probabilities equals one) are satisfied. After normalization, the sampled values are then used in a traditional AFA [either Eqs. (2–5) or Eqs. (8–11)] to obtain an adjusted model, referred to as y_{adj}^j. This process of sampling the model probability distributions is then repeated n times, resulting in a set of adjusted models $\{y_{\mathrm{adj}}^1, y_{\mathrm{adj}}^2, \ldots, y_{\mathrm{adj}}^n\}$, with each adjusted model representing the result of a different set of model probabilities. These individual adjusted models are then sampled using Markov chain Monte Carlo sampling, using a Metropolis chain [17] with m samples. Using the m samples of the n adjusted models, a new aggregate adjusted model y_{maf} can then be constructed using the statistical properties of the m samples and the same assumption regarding the form of the adjusted model as was used in the AFA.

After completing the modified AFA, two adjusted models now exist that represent the model-form uncertainty in the problem of interest: y, which uses the deterministic model probability values obtained from expert opinions, and y_{maf}, which represents the potential variance in the prediction of the adjusted model as a result of perturbations of $P(M_i)$. A metric must now be implemented that measures the similarity of the two models. The Bhattacharyya distance is a metric developed to measure the geometric similarity between two distinct distributions [Eq. (13)] [18]:

$$BC(f_{x_1}, f_{x_2}) = \int_{-\infty}^{\infty} f_{x_1}(x)^{0.5} f_{x_2}(x)^{0.5}\,\mathrm{d}x \tag{13}$$

In Eq. (13), $f_{x_1}(x)$ and $f_{x_2}(x)$ represent the distributions of the two models of interest, y and y_{maf}, respectively. The Bhattacharyya distance is a bounded value between zero and one, where a value of one implies that the two models of interest are identically distributed. As such, a value of the Bhattacharyya distance close to one implies a greater similarity between the two models being considered, whereas a lower value implies that there is a greater variance between the models.

B. Predictive Uncertainty

While model-form uncertainty denotes the discrepancies between multiple models of interest, predictive uncertainty denotes the difference between a model and the true physical scenario that is being represented in the model [1]. The presence of predictive uncertainty is a direct result of the simplifying assumptions made in the construction of a model, such as an inviscid or incompressible flow assumption in an aerodynamic analysis. As a result, each individual model has its own unique predictive uncertainty associated with it, as the assumptions in each model are not necessarily the same.

To quantify predictive uncertainty, information regarding the true physical scenario of interest must be known. While this information that is acquired is commonly experimental data points, a couple of caveats are introduced. The experimental data that are acquired are not necessarily infallible for a couple of reasons. First, experiments are often done on a reduced-order model or, at least, a model with artificial constraints, such as a full-scale wing test in a wind tunnel. As such, the experiment could technically be considered an additional model. In addition, measuring the value of parameters or outputs in an experiment is an imprecise science. As such, there is additional uncertainty introduced to the problem, in that the supposed true value is also an uncertain value. To represent the complete picture of uncertainty, Kennedy and O'Hagan represent the true physical scenario as shown in Eq. (14) [19]:

$$d_k = \rho Pr(M_i|\theta_k) + \delta(y_i) + \varepsilon_{\mathrm{exp}} \tag{14}$$

In Eq. (14), d_k represents the true physical value of an output, such as flutter velocity, that is trying to be represented by the models of interest. The equation shows that this true value is actually the sum of three different terms. In the first term, ρ represents an unknown regression parameter that cannot be solved for empirically. $Pr(M_i|\theta_k)$ represents the results of model i given the parameter set θ_k. This could either be a probability distribution, as denoted by the equation, or a deterministic value if the model is deterministic in nature. The discrepancy term in the equation that represents the difference between the model result and the true physical scenario is $\delta(y_i)$. This function is also referred to as the model inadequacy function [19], and it is independent of the model output $Pr(M_i|\theta_k)$. Finally, $\varepsilon_{\mathrm{exp}}$ is the observation error term, and it represents the uncertainty that exists in the measurement of output d_k. This term can either be a deterministic value or a distribution representing the uncertainty in the measurement of the experimental value. By rearranging Eq. (14), expressions for different representations of the predictive uncertainty for a modeling problem can be developed, as will be shown in the following section.

1. Bayesian Model Averaging

BMA [3] is a methodology that quantifies both the model-form uncertainty addressed before as well as the predictive uncertainty introduced here. Although the models can be individually considered deterministic, the presence of predictive uncertainty dictates that they should instead be described as distributions to account for the known errors as a result of simplifying assumptions. To account for the errors that exist in the simulation of a physical scenario through a model, an assumption is made that each model's estimation contains a residual that is identically, independently, and normally distributed (IDD). As such, by renaming the discrepancy term $\delta(y_i)$ as ε_{ik}, the new term can be solved for as a function of the difference between the experimental data point and a particular model's prediction of the value of that quantity, as shown in Eq. (15) [20]:

$$\varepsilon_{ik} = d_k - \rho Pr(y|M_i, \theta_k) \sim^{\mathrm{IDD}} N(0, \sigma_k) \tag{15}$$

In Eq. (15), d_k represents the kth experimental result of the experimental data point set D at the vector of input variables x_k, while $Pr(y|M_i, d_k)$ represents the model i solution at the same design variable vector. Although, in this case, the experimental data point is represented as a deterministic value d_k, Eq. (15) can be rederived from the Kennedy equation of model error [Eq. (14)] to handle uncertainties in the measurement of the experimental values. However, in the scope of this work, all experimental values will be assumed to be deterministic. Finally, σ_k represents the standard deviation of the normal distribution, as calculated by Eq. (16):

$$\sigma_k = \sqrt{\frac{\sum_{k=1}^{m} \varepsilon_{ik}^2}{m}} \tag{16}$$

Now that the residual of each model is defined, the predictive distribution of each of the models can be constructed by adding the residual to the deterministic model prediction, as shown in Eq. (17):

$$Pr(y|M_i, D) = Pr(y|M_i) + \varepsilon_{ik}$$
$$\forall\ i, k = \text{normal}\ \{E[Pr(y|M_i)], \sigma_k^2\} \tag{17}$$

After forming stochastic model predictions for each of the models, BMA can be applied to calculate the adjusted model given the experimental data set D [Eq. (18)]:

$$Pr(y|D) = \sum_{i=1}^{n} Pr(M_i|D) Pr(y|M_i, D) \tag{18}$$

with an expected value and variance calculated in Eqs. (19) and (20):

$$E[Pr(y|D)] = \sum_{i=1}^{n} Pr(M_i|D) E[Pr(y|M_i, D)] \tag{19}$$

$$\mathrm{Var}[Pr(y|D)] = \sum_{i=1}^{n} Pr(M_i|D) \mathrm{Var}[Pr(y|M_i, D)]^2 + \sum_{i=1}^{n} Pr(M_i|D)\{E[Pr(y|M_i, D)] - E[Pr(y|D)]\}^2 \tag{20}$$

In Eqs. (18–20), $Pr(M_i|D)$ represents model probability, which is similar to the $P(M_i)$ used in the AFA. However, unlike the AFA, where expert opinion was used to determine the values for model probability, BMA assumes, without the presence of additional data, that all model probabilities are equal such that Eq. (21) is still satisfied:

$$\sum_{i=1}^{n} Pr(M_i|D) = 1 \tag{21}$$

2. Bayesian Model Updating

Although BMA initially assumes equal probability among all models, by including additional experimental data points in the analysis, these model probabilities can be updated and refined to include the additional knowledge introduced to the problem. This updating of model probabilities can be done by Bayesian model updating. By using experimental data points, the updated model probabilities for each of the models can be calculated by Eq. (22):

$$Pr(M_i|D) = \frac{Pr(M_i) \times Pr(D|M_i)}{\sum_{j=1}^{N} Pr(M_j) \times Pr(D|M_j)} \tag{22}$$

where $Pr(M_i)$ represents the probability of model M_i before the observation of experimental data and $Pr(D|M_i)$ represents the likelihood of model M_i given an experimental data set D, which can be calculated as shown in Eq. (23):

$$Pr(D|M_i) = \int Pr(D|\bar{x}, M_i) Pr(\bar{x}|M_i)\, \mathrm{d}\bar{x} \tag{23}$$

This updating process can then be repeated for each of the additional data points to provide updated model probabilities to the BMA approach.

C. Modeling Uncertainty Quantification Framework

While approaches such as BMA are very powerful and capable of quantifying model-form and predictive uncertainties, they also have a necessity for experimental data points, which are not always readily available. The limited availability of these data points is even more present in the preliminary design phase, where numerous design configurations are often considered concurrently. Thus, it is

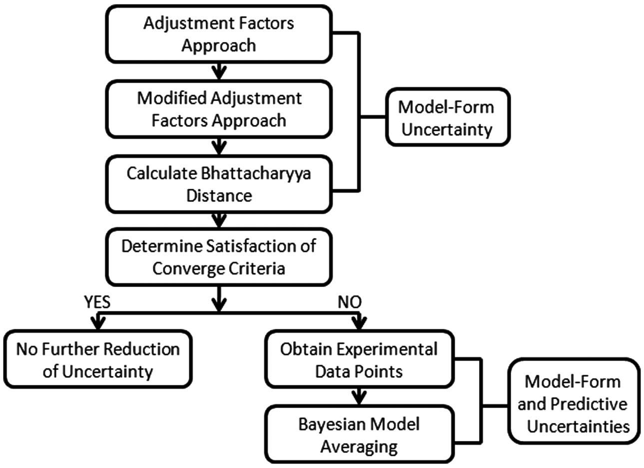

Fig. 2 Modeling uncertainty framework.

infeasible to assume that experimental data points will always be readily available for quantification of model-form and predictive uncertainties. Similarly, it would be inefficient and cost restrictive to obtain experimental data points at each stage of the preliminary design phase to quantify this uncertainty. To efficiently solve this problem, a framework has been proposed in this work that uses the model-form uncertainty present in the problem at the time to determine the necessity of additional experimental data points (Fig. 2).

Within this framework, the model-form uncertainty in the problem is initially quantified using the AFA. In this approach, the model probabilities are assigned using expert opinions regarding the accuracy and robustness of the models in consideration. Next, the modified AFA is used to calculate the sensitivity of the adjusted model's response to the individual model probabilities. As a result of these two approaches, two different adjusted models will be created: y and y_{maf}. The Bhattacharyya distance will then be calculated between these two distributions to determine the similarity of the adjusted models. If the Bhattacharyya distance is shown to be less than a critical value, denoting a significant variance between the two adjusted models, then the adjusted model y can be considered sensitive to the model probabilities assigned through expert opinions. In this work, a value less than 0.99 is deemed to be critical to denote sensitivity of the adjusted model to the individual model probabilities. The selection of a critical value is a very important design decision that is dependent on many factors, such as the cost of an additional data point and the number of models being considered. In this work, a suitable value was determined and used for the analysis, but a rigorous method for determining a proper critical value will be addressed in future work. The model-form uncertainty in the problem could then be further reduced through the introduction of experimental data points. In addition, the introduction of these data points will allow for the quantification of the predictive uncertainty in the problem. This further quantification will then be performed using BMA, with experimental data points now obtained for the problem.

By using a framework such as the one proposed in this work, the modeling uncertainty in the problem can still be quantified and reduced to an acceptable amount at minimal experimental cost. Instead of performing a blanket number of experiments at various data points and configurations, or even using traditional design of experiments methodology, this framework uses the modeling uncertainty itself to drive the necessity of further experimental data points.

III. Modeling Uncertainty in Two-Degree-of-Freedom Flutter Problem with Unsteady Aerodynamics

To demonstrate the application of the modeling uncertainty framework to a simple aeroelastic problem, the flutter velocity of a 2-DOF (pitching and plunging) airfoil subject to unsteady aerodynamics (Fig. 3) will be solved with the parameters detailed in Table 1.

Including the presence of circulatory flow, the lift and the moment about the shear can be calculated by Eqs. (24) and (22), respectively:

$$L_{SC} = \pi\rho b^2[\ddot{h} - U\dot{\alpha} - ba\ddot{\alpha}] + 2\pi\rho UbC(k)[\dot{h} + U\alpha + b(\tfrac{1}{2} - a)\dot{\alpha}] \tag{24}$$

$$M_{SC} = \pi\rho b^2[\beta a\ddot{h} - Ub(\tfrac{1}{2} - a)\dot{\alpha} - b^2(\tfrac{1}{8} + a^2)\ddot{\alpha}] + 2\pi\rho Ub^2(a + \tfrac{1}{2})C(k)[\dot{h} + U\alpha + b(\tfrac{1}{2} - a)\dot{\alpha}] \tag{25}$$

In the preceding equations, $C(k)$ represents the Theodorsen circulation function, which controls the phasing and amplitude of the lift and pitching moments with respect to the airfoil motion. The Theodorsen circulation function is a complex function consisting of both real and imaginary parts. Multiple surrogate functions exist in the literature to approximate the Theodorsen function as a function of k over the range of k values experienced by the system [Eqs. (26–31)]:

$$C_1(k) = 1 - \frac{0.165k}{k - 0.0455i} - \frac{0.355k}{k - 0.3i} \tag{26}$$

$$C_2(k) = \frac{0.01365 + 0.2808ik - 0.5k^2}{0.01365 + 0.3455ik - k^2} \tag{27}$$

$$C_3(k) = \frac{J_1 - J_{oy}i}{J_1 - J_{oy}i + J_o - J_{1y}i} \tag{28}$$

$$C_4(k) = \frac{(1 + 10.61ik)(1 + 1.774ik)}{(1 + 13.51ik)(1 + 2.745ik)} \tag{29}$$

$$C_5(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + iH_0^{(2)}(k)} \tag{30}$$

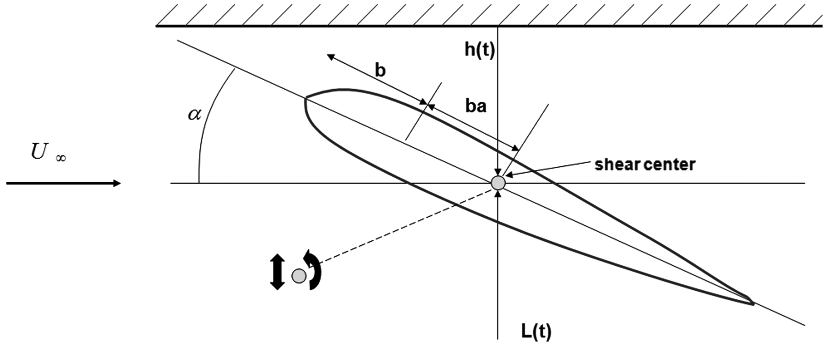

Fig. 3 Two-DOF wing.

Table 1 Airfoil parameters

Parameter	Value
μ	20
ω_h	10 rad/s
ω_θ	25 rad/s
b	36 in.
a	−0.20

$$C_6(k) = \frac{0.015 + 0.3ik - 0.5k^2}{0.015 + 0.35ik - k^2} \tag{31}$$

where k is the reduced frequency of the system defined by Eq. (32), where ω is the frequency of oscillation of the airfoil and U_∞ is the freestream velocity:

$$k = \frac{\omega \cdot b}{U_\infty} \tag{32}$$

The real and imaginary part of these six surrogate models vary over an average operating range of k, as shown in Fig. 4.

Once a surrogate model of the Theodorsen function is selected, the flutter velocity for this system of equations can then be solved using the theory of unsteady aerodynamics [Eqs. (25–27)] and the V-g solution method. However, due to multiple possible surrogate models, as well as the fact that each of the surrogate models is not representative of the exact table of Theodorsen values, there is inherent modeling uncertainty to the problem. Solving for the flutter velocity of the airfoil with each of the six Theodorsen function approximations [Eqs. (26–31)] produced results that can be seen in Table 2.

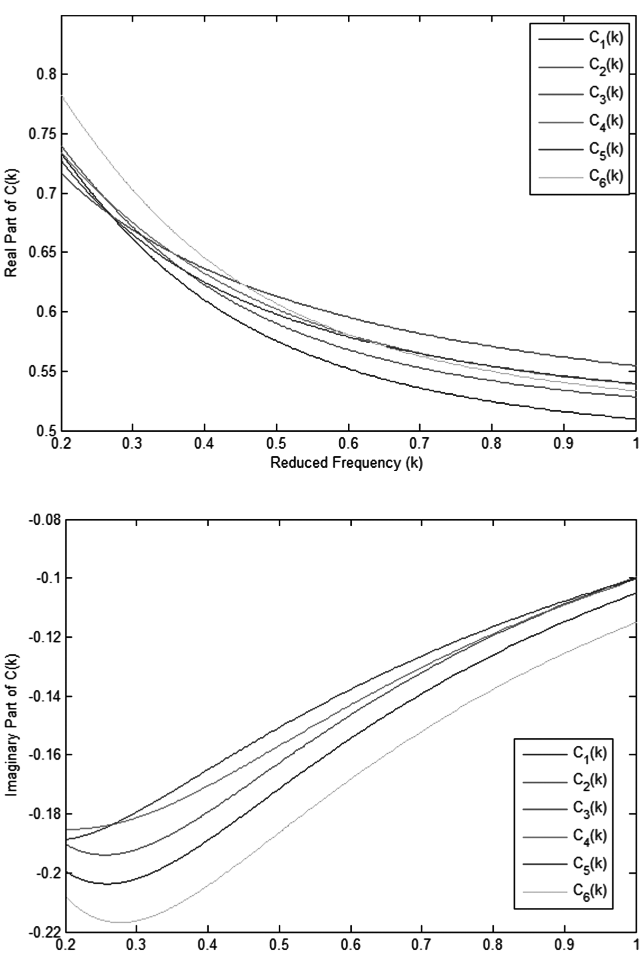

Fig. 4 Real and imaginary components of $C(k)$ for six models.

Table 2 Flutter velocities and model probabilities for six models

	V_f, ft/s	$P(M_i)_{\text{prior}}$
$C_1(k)$	101.39	0.20
$C_2(k)$	99.47	0.10
$C_3(k)$	97.97	0.25
$C_4(k)$	97.60	0.10
$C_5(k)$	98.30	0.30
$C_6(k)$	92.11	0.05

By assigning a probability to each of the six models based on expert opinion, an AFA can be used to develop a distribution of the flutter velocity for the 2-DOF system. To determine which AFA to use, prior knowledge regarding a final model prediction is used. However, in the absence of such prior knowledge, a normal distribution was assumed, and the additive AFA was used. Table 3 shows the means and standard deviations of the figure of merit of the adjusted model for the flutter velocity of the system developed via the additive AFA. The adjusted model y is then plotted in Fig. 5, showing the probability density function (PDF) for the flutter velocity of the wing given the model-form uncertainty induced by the multiple model results.

The next step in the modeling uncertainty framework is to attempt to quantify the model uncertainty in the problem or, more specifically, the uncertainty associated with the probabilities assigned to each of the models. By using the modified AFA developed in this research, the model probabilities $P(M_i)$ can be assigned a distribution, and the effect of their inherent uncertainty can be determined. By defining each of the six model probabilities as a normal distribution with the mean of their deterministic value (shown in Table 2) and a standard distribution as defined in Eq. (12), the Bhattacharyya measure for the two models can be calculated, as shown in Table 3.

Looking at the results shown in Table 3, it can be seen that the Bhattacharyya measure for the two models of interest is 0.98009, which is less than the threshold value of 0.99 used for this work. As a result, the adjusted model is shown to be sensitive to the model probabilities shown in Table 2. As such, the next step in the modeling uncertainty framework is to obtain experimental data points for use in BMA.

As mentioned previously, the BMA approach requires experimental data points for implementation. In the case of the problem being presented, a wind-tunnel experimental data point was available in the literature for the airfoil being analyzed. The experimental data point for the problem of interest is a measured flutter velocity of 98 ft/s [21]. While the framework is capable of handling uncertainties in the measurement of this experimental data, the wind-tunnel data provided in this work are deterministic. For cases where

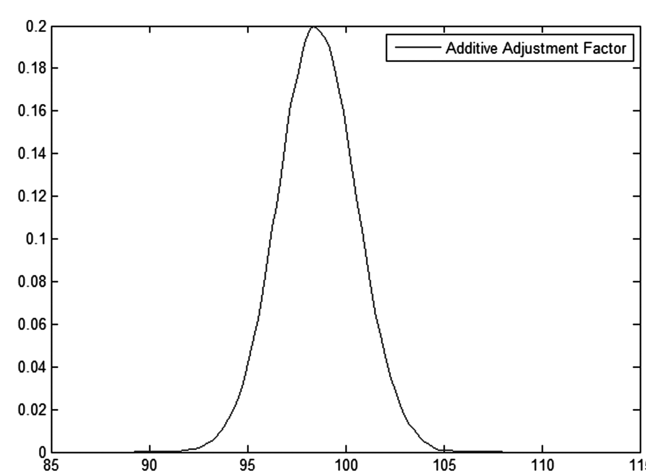

Fig. 5 PDF of adjusted model y.

Table 3 Modified AFA result

	Mean, ft/s	Standard deviation
Additive AFA	95.57	1.99
Modified AFA	98.37	2.63
Model disagreement	0.2%	32.2%
Bhattacharyya measure		0.98009

Table 4 Updated model probabilities

	V_f, ft/s	$P(M_i)_{\text{prior}}$	$P(M_i)_{\text{post}}$
$C_1(k)$	101.39	0.1666	0.0077
$C_2(k)$	99.47	0.1666	0.0178
$C_3(k)$	97.97	0.1666	0.8180
$C_4(k)$	97.60	0.1666	0.0651
$C_5(k)$	98.30	0.1666	0.0870
$C_6(k)$	92.11	0.1666	0.0044

Table 5 Adjusted models from two approaches

	Mean, ft/s	Standard deviation
AFA	98.57	1.99
BMA	97.48	0.88

there is established uncertainty in the experimental data results, such as a confidence bound, the data point is then represented as a distribution and propagated through Eq. (14) to incorporate the uncertainty with the experimental data point(s). As such, its implementation within this framework is as a deterministic value. The framework then uses Bayesian model updating to produce updated model probabilities. While expert opinion was used to produce biased model probabilities in the AFA, a foundation of the Bayes theory is that, in the absence of any physical data regarding the system, all model probabilities must be initially assumed to be equal [3]. By using Eq. (22), the model probabilities can be updated, as the results show in Table 4.

The updated model probabilities, along with the new experimental data point, are then used in the BMA approach to develop a new adjusted model for the flutter velocity of the system that takes into account both predictive and model-form uncertainties. The parameters for this model can be seen in Table 5 and are plotted in Fig. 6.

Figure 6 illustrates that, by introducing the additional knowledge of the experimental data, two primary changes in the adjusted model are noted. First, the mean of the adjusted model is shifted lower, dropping to about 98 ft/s. This value for the mean is logical, as the experimental data point occurred at 98 ft/s. Using this data in Bayesian model updating, the model probabilities were weighted to include this information. In addition, the BMA approach uses this experimental data point in addition to the individual model predictions to form the adjusted model. The second change of note is the reduction in the variance of the adjusted model. The variance in the adjusted model produced by the BMA approach is less than half of the variance produced by the original AFA. The reduction in this variance is due to the inclusion of additional information (in this case, the experimental data point) used in the quantification of the variance between the models.

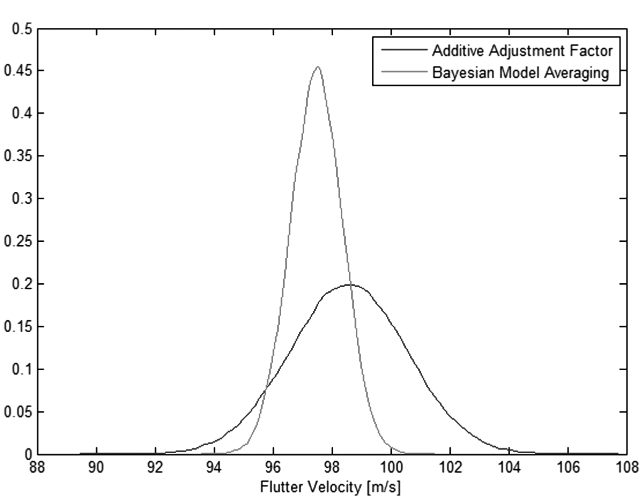

Fig. 6 PDFs for AFA and BMA adjusted models.

IV. Conclusions

In this paper, a methodology was presented for the quantification of modeling uncertainty. While prior uncertainty quantification techniques concentrate solely upon the quantification of parametric uncertainty, this work introduces methods for quantifying two additional forms of uncertainty in modeling problems: model-form and predictive uncertainties. A framework was presented that uses Bayesian approaches to quantify the model-form uncertainty in a problem with disagreement among multiple models. In the initial stages of modeling uncertainty quantification, the framework uses lower-fidelity model-form uncertainty techniques, such as the AFA, to obtain an estimate of the degree of model-form uncertainty in the problem using only the results of individual models and expert opinions regarding the relative merit and accuracy of those models. Then, using the newly developed modified AFA, the sensitivity of the adjusted model to the model probabilities formed from expert opinions is estimated, providing an initial converge parameter for the modeling uncertainty quantification framework. Should this criteria be met, the uncertainty induced by disagreement between the models is shown to be accurately represented by the adjusted model produced by the AFA, and further refinement of this model could only be done through the modification of the individual models themselves. However, should this criteria not be met, it would indicate that the adjusted model is sensitive to the individual model probabilities used in the problem and that further analysis of the models is necessary to fully reduce the modeling uncertainty in the problem. This framework would then call for the introduction of experimental data points, which would be used to reduce the model-form uncertainty in the problem, and quantify the predictive uncertainty associated with the model set. By using this framework, experimental data points, which are often very costly to the designer in the preliminary design phase, can be used in the regions where they would be most beneficial to the reduction of the modeling uncertainty in the problem rather than requiring experimental data points at every step along the way.

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